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NOTE ON MULTIPLY PERFECT NUMBERS.*

BY JACOB WESTLUND.

IN the ANNALS OF MATHEMATICS, ser. 2, vol. 2 (Jan. 1901), p. 103, Dr. D. N. Lehmer proves that no multiply perfect numbers of multiplicity 3, containing less than three distinct primes, exist. The object of the present note is to determine all numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2^{a_2} p_3$ where p_1, p_2, p_3 are three distinct primes and $p_1 < p_2 < p_3$.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity, we have†

$$(1) \qquad 3 = \frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)} \cdot \frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)} \cdot \frac{p_3^2-1}{p_3(p_3-1)}.$$

In order to obtain necessary conditions which p_1 and p_2 must satisfy, we use the inequality†

$$(2) \qquad 3 < \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1} \cdot \frac{p_3}{p_3-1}.$$

From this we infer that p_1 and p_2 must be 2 and 3 respectively, for the maximum value of $\prod_{n=1}^{3} \frac{p_n}{p_n-1}$ will exceed 3 only for $p_1 = 2$ and $p_2 = 3$. Hence we have

$$(3) \qquad 3 = \frac{2^{a_1+1}-1}{2^{a_1}} \cdot \frac{3^{a_2+1}-1}{3^{a_2} \cdot 2} \cdot \frac{p_3+1}{p_3},$$

from which we get

$$(4) \qquad p_3 + 1 = \frac{2^{a_1+1}\,3^{a_2+1}}{2^{a_1+1} + 3^{a_2+1} - 1}.$$

Hence the only prime factors that $p_3 + 1$ can have are 2 and 3:

$$p_3 + 1 = 2^{\mu} 3^{\nu}. \qquad (0 < \mu < a_1 + 1, \quad 0 \leqq \nu \leqq a_2 + 1).$$

Here, μ cannot have the value 0, since $p_3 + 1$ is an even number. From (3) we get

$$p_3\, 2^{a_1+1}\, 3^{a_2+1} = (2^{a_1+1} - 1)(3^{a_2+1} - 1)(p_3 + 1),$$

or

$$(5) \qquad p_3\, 2^{a_1+1-\mu}\, 3^{a_2+1-\nu} = (2^{a_1+1} - 1)(3^{a_2+1} - 1).$$

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* Read before the Chicago Section of the American Mathematical Society, April 6, 1901.

† *cf.* Lehmer, *l. c.*

Since the parentheses in the right hand member of this equation are prime respectively to 2 and 3, two possible cases present themselves. We must have either

$$\text{I.}\quad \begin{cases} 3^{a_2+1}-1 = p_3\,2^{a_1+1-\mu} = 2^{a_1+1}\,3^{\nu} - 2^{a_1+1-\mu} \\ 2^{a_1+1}-1 = 3^{a_2+1-\nu} \end{cases}$$

or

$$\text{II.}\quad \begin{cases} 3^{a_2+1}-1 = 2^{a_1+1-\mu} \\ 2^{a_1+1}-1 = p_3\,3^{a_2+1-\nu} = 2^{\mu}\,3^{a_2+1} - 3^{a_2+1-\nu} \end{cases}$$

Case I. In this case we have

$$2^{a_1+1}-1 = 3^{a_2+1-\nu}.$$

Let

$$a_1+1 = 2^{\lambda}(2n+1).$$

Then

$$2^{a_1+1}-1 = 2^{2^{\lambda}(2n+1)}-1.$$

1. If $\lambda = 0$, then $2^{a_1+1}-1 = 2\cdot 4^n - 1 = 2(4^n-1)+1$ which is not divisible by 3, since (4^n-1) is divisible by 3.
2. If $\lambda = 1$, then $2^{a+1}-1 = (2^{2n+1}-1)(2^{2n+1}+1)$, which contains other prime factors besides 3, unless $n = 0$; since if one of the factors is divisible by 3, the other is not. For $n = 0$ we have $a_1+1 = 2$ and $a_2+1 = \nu+1$ and hence

$$3^{\nu+1}-1 = 2^2\,3^{\nu} - 2^{2-\mu},$$

or

$$3^{\nu} = 2^{2-\mu}-1.$$

This gives $\mu = 1$, $\nu = 0$ which is impossible since $p_3 > 3$.

3. If $\lambda > 1$, then $2^{a_1+1}-1 = 16^{2^{\lambda-2}(2n+1)}-1$, which is always divisible by 5.

Thus in Case I there are no multiply perfect numbers of the type here considered.

Case II. In this case we have

$$3^{a_2+1}-1 = 2^{a_1+1-\mu}.$$

Let

$$a_2+1 = 2^{\lambda}(2n+1).$$

Then

$$3^{a_2+1}-1 = 3^{2^{\lambda}(2n+1)}-1.$$

1. If $\lambda = 0$, then $3^{a_2+1}-1 = 3(9^n-1)+2$, which contains other primes besides 2 unless $n = 0$. For $n = 0$ we have $a_2+1 = 1$ which is impossible.

2. If $\lambda = 1$, then $3^{a_2+1} - 1 = 9(81^n - 1) + 8$, which contains other primes besides 2 unless $n = 0$. For $n = 0$ we have $a_2 + 1 = 2$ and $a_1 + 1 = \mu + 3$. Hence

$$2^{\mu+3} - 1 = 2^{\mu} \cdot 3^2 - 3^{2-\nu}$$

or

$$2^{\mu} = 3^{2-\nu} - 1.$$

The only values of μ and ν which satisfy this equation are $\mu = 3$, $\nu = 0$ and $\mu = 1$, $\nu = 1$. The corresponding values of a_1, a_2, p_3 are $a_1 = 5$, $a_2 = 1$, $p_3 = 7$ and $a_1 = 3$, $a_2 = 1$, $p_3 = 5$.

It is found by trial (Lehmer, *l. c.*) that the resulting numbers, $2^5 \cdot 3 \cdot 7$ and $2^3 \cdot 3 \cdot 5$ are multiply perfect.

3. If $\lambda > 1$, then $3^{a_2+1} - 1 = 81^{2^{\lambda-2}(2n+1)} - 1$, which is divisible by 5.

Hence the only multiply perfect numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2^{a_2} p_3$ are the two numbers $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$.

PURDUE UNIVERSITY, FEBRUARY, 1901.